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Antonie Kotzé and Angelo Joseph

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The exchange traded index option market in South Africa has seen tremendous growth during the last couple of years. The biggest liquidity is in options on the near and middle Alsi future contracts. Alsi futures are listed future contracts on the FTSE/JSE Top40 index, the most important and tradable equity index in South Africa. OTC and listed options trade on a skew and most market makers have implemented their own proprietary skew generators. Clearing houses also use the volatility surface in estimating the initial margins for options.

In this paper we show how to generate the implied volatility surface by fitting a quadratic deterministic function to implied volatility data from Alsi index options traded on Safex. This market is mostly driven by structured spread trades, and very few at-the-money options ever trade. It is thus difficult to obtain the correct at-the-money volatilities needed by the exchange for their mark-to-market and risk management processes. We further investigate the term structure of at-the-money volatilities and show how the at-the-money implied volatilities can be obtained from the same deterministic model. This methodology leads to a no-spread arbitrage and robust market related volatility surface that can be used by option traders and brokers in pricing structured option trades.



# Constructing a South African Implied Volatility Surface from Exchange Traded Data

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ANTONIE KOTZÉ AND ANGELO JOSEPH<sup>†</sup>

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In this paper we show how to generate the implied volatility surface by fitting a quadratic deterministic function to implied volatility data from Alsi index options traded on Safex<sup>1</sup>. This market is mostly driven by structured spread trades, and very few at-the-money options ever trade. It is thus difficult to obtain the correct at-the-money volatilities needed by the exchange for their mark-to-market and risk management processes. We further investigate the term structure of at-the-money volatilities and show how the at-the-money implied volatilities can be obtained from the same deterministic model. This methodology leads to a no-spread arbitrage and robust market related volatility surface that can be used by option traders and brokers in pricing structured option trades.

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<sup>†</sup>Consultants from Financial Chaos Theory — experts on modeling the complexity of financial derivatives in a developing markets context. For further information surf to <http://www.quantonline.co.za> or email [consultant@quantonline.co.za](mailto:consultant@quantonline.co.za).

<sup>1</sup>The South African derivatives exchange based in Johannesburg — <http://www.safex.co.za/ed/>

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# 1 Introduction

Traders say volatilities are “skewed” when options of a given asset trade at increasing or decreasing levels of implied volatility as you move through the strikes. The volatility skew (smile) was first observed and mentioned by Black and Scholes in a paper that appeared in 1972 [BS 72]. It was then empirically described in 1979 by Macbeth and Merville [MM 79]. At that point in time it wasn’t pronounced but the market crash of October 1987 changed all of that.

If one studies option prices before and after October 1987, one will see a distinct break. Option prices begin to reflect an “option risk premium” — a crash premium that comes from the experiences traders had in October 1987. After the crash, the demand for protection rose and that lifted the prices for puts; especially out-the-money puts. To afford protection, investors would sell out-the-money calls. There is thus an over supply of right hand sided calls and demand for left hand sided puts — alas the skew. In fact, out-the-money put options usually trade at a premium to out-the-money calls.

The market also observed that near dated at-the-money options usually trade at higher volatilities than far dated options. This is called the at-the-money term structure of volatility, reflecting the effect that volatility reverts to a long range mean; volatility is mean reverting. A model that generates a volatility surface from traded option data must be able to capture these stylised facts. If so, we will obtain reliable valuations and sound risk measures. An accurate volatility surface is also very important to futures clearing houses. The margin requirements for options are based on the volatility surface. If the surface is not accurate or not reflective of the market, it will result in margin requirements that might not protect the clearing house against the current risk in the market.

The smile phenomenon has spread to commodity options, interest-rate options, currency options and almost every other option market. Since the Black-Scholes model cannot account for the skew, option traders and risk managers began using more complex models to value and manage options. Some of these methods include generating a market related implied volatility surface that accurately reflects the correct option values.

In this paper we focus on two important aspects of volatility: we firstly introduce a simple quadratic deterministic volatility function that we fit to traded data giving a robust traded implied volatility surface (IVS). Secondly (and independent from the first step), we use a similar function to obtain the fair at-the-money (ATM) term structure of volatility (TSV).

An aspect that always rears its head in developing markets is the impact of market efficiency i.e., liquidity. In South Africa, there are very few ATM index option trades. The market mostly trade spread structures with strikes not too far from the ATM. This means that deep in-the-money (ITM) and deep out-the-money (OTM) contracts rarely trades. This has an impact on the accuracy of volatilities in the wings. We show how to take all of these aspects into consideration when generating the IVS and

ATM term structure of volatility.

This paper is organised as follows: in §2 we look at the general dynamics of volatility setting the tone for the complexities in modelling volatility. In §4 we briefly introduce the stochastic and non-parametric methods used to capture the dynamics of volatility. Then, in §5 we delve into the deterministic approaches. We look at the history of this approach and give some reasons for why this methodology is suitable to the South African market. In this part of the paper we introduce the quadratic function used to generate the volatility surface and also the decaying function used to generate the at-the-money term structure of volatility. In §6 we discuss the implementation and optimisation of the quadratic and term structure functions. Full details of the optimisations are given in some Appendixes.

## 2 Volatility Dynamics

The Black & Scholes option pricing model assumes that volatility is constant [Ko 03]. However, a very interesting working paper by David Shimko reported very high negative correlations during the period 1987-1989 between changes in implied volatilities on S&P 100 index options and the concurrent return of the index — a correlation which should be zero according to the Black-Scholes formula [Sh 91].

It is now an accepted fact<sup>2</sup>, that when equity prices go up (down) volatility usually goes down (up); there is an inverse relationship between volatility and the underlying asset price. We show this in Figure 1 where we plot the FTSE/JSE Top40 index together with the 1 month historical volatility. The Black & Scholes option model with constant volatility will therefore produce option prices that do not match those traded in the market. This deficiency of the Black & Scholes model arises because Black & Scholes assumed a constant volatility in deriving their option pricing formula [Ko 02]. It is also argued that the source of this Black-Scholes deficiency can be attributed to the fact that the distribution of equity price levels at expiry is not lognormal [DFW 98].

But what is “volatility”? A price series or an economic indicator that changes a lot and swings wildly is said to be “volatile”. This simple and intuitive concept is the cause of many difficulties in finance. Unlike many other market parameters which can be observed directly, volatility has to be estimated. This is difficult, if not impossible, because we cannot say that volatility is necessarily stochastic or that it conforms to any mathematical model. All we know is that the evolution of volatility is uncertain. An accurate estimate of volatility is, however, crucial in many applications, including risk measurement and management as well as option pricing and hedging [Ko 01].

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<sup>2</sup>Even Black wrote in 1976 “I have believed for a long time that stock returns are related to volatility changes. When stocks go up, volatility seems to go down; and when stocks go down, volatility seems to go up.” ([Bl 76], p. 177)

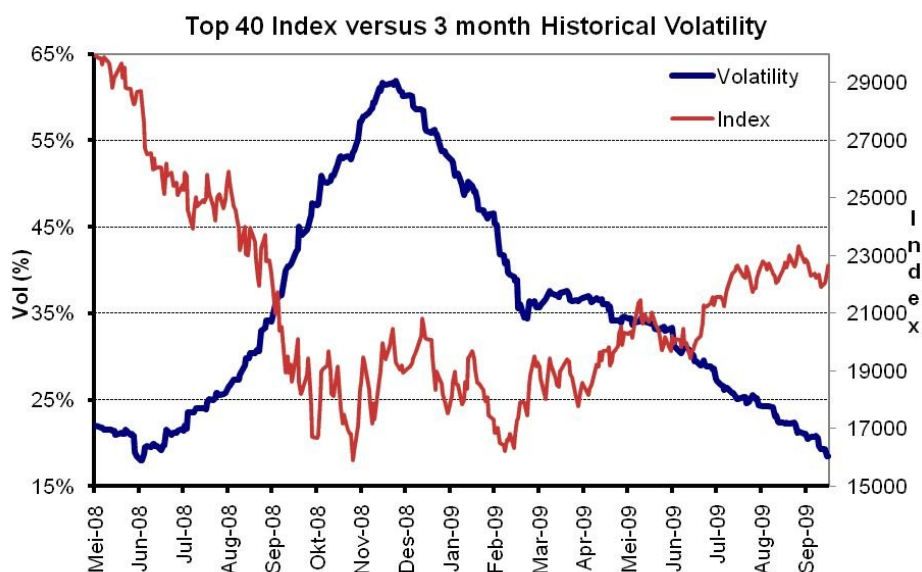


Figure 1: FTSE/JSE Top40 index level and the 3 month historical volatility during the period, June 2008 through October 2009. The plot shows the negative correlation that usually exists between the index level and the volatility.

### 3 Margin Requirements by Clearing Houses

Margining is an important part of the risk management process utilised by an exchange. To minimize risk to the exchange, derivative traders must post margin. Margin helps derivative exchanges to avoid credit and market risk, i.e. the chance of one or more counterparties to a trade, defaulting on their obligations. They accomplish this in two ways. Firstly, all trades on an exchange are settled or “cleared” through a clearinghouse which may be a separate legal entity to the exchange itself. The JSE’s clearing house is SAFCOM. The clearinghouse acts as the principal counterparty to all trades through an exchange. Thus, it interposes itself as the ‘buyer to every seller’ and the ‘seller to every buyer’ — known as novation. Through novation Safcom guarantees to its members the financial performance of all contracts traded. SAFCOM becomes the guarantor of all derivative transactions allowing members participants to deal freely with each other without counterparty credit risk constraints.

Secondly, exchanges employ a system of margining. The exchange then also estimates what losses are possible in the future — usually 1 trading day. Participants are required to lodge margins with the exchange which are sufficient to cover these possible future losses — this is called initial margin. Should the losses eventuate and the participant be unable to bear them, the margin is available to the exchange to meet the shortfall. The initial margin may be reduced or increased based on changes in the margin parameters. There are two stages to estimating possible future losses and the initial margin requirements:

- The exchange does a statistical analysis of historical market moves and subjective assessments of the state of the market. They express the maximum anticipated price and volatility moves between the present and the next mark-to-market day. This is similar to a Value at Risk (VAR) calculation.
- Secondly, the exchange re-values each position at this maximum anticipated price and volatility at the next mark-to-market day. The margin covers this maximum conceivable mark-to-market loss that the position (entire portfolio) could suffer.

Initial margin requirements for options are thus directly linked to the volatility surface. An accurate market surface will lead to initial margins that reflect the current risks in the market. An inaccurate or stale surface impose “unaccounted” risks onto the clearing house. Having an accurate market related implied volatility surface is extremely important to a clearing house’s risk management processes.

## 4 Stochastic and Nonparametric Volatility Models

The idea that the price of a financial instrument might be arrived at using a complex mathematical formula is relatively new. This idea can be traced back to the seminal paper by Myron Scholes and Fischer Black [BS 73]. We now live in a world where it is accepted that the value of certain illiquid derivative securities can be arrived at on the basis of a model (this is the practise of marking-to-model) [Re 06].

In order to implement these models, practitioners paid more and more attention to, and began to collect, direct empirical market data often at a transactional level. The availability of this data created new opportunities. The reasonableness of a model’s assumptions could be assessed and the data guided many practitioners in the development of new models. Such market data led researchers to models whereby the dynamics of volatility could be studied and modeled.

Stochastic, empirical, nonparametric and deterministic models have been studied extensively. In this section we will give a brief overview of the first 3 models before we delf into the deterministic model in greater detail in §5.

### 4.1 Stochastic Models

Black & Scholes defined volatility as the standard deviation because it measures the variability in the returns of the underlying asset [BS 72]. They determined the historical volatility and used that as a proxy for the expected or implied volatility in the future. Since then the study of implied volatility has become a central preoccupation for both academics and practitioners [Ga 06]. Volatility changes over time and seem to be driven by a stochastic process. This can be seen in Figure 2 where we plot the 3 month rolling volatility for the FTSE/JSE Top40 index. Also evident from this plot



is the mean reversion phenomenon. The 3 month long term mean volatility of this index is 21.38%.

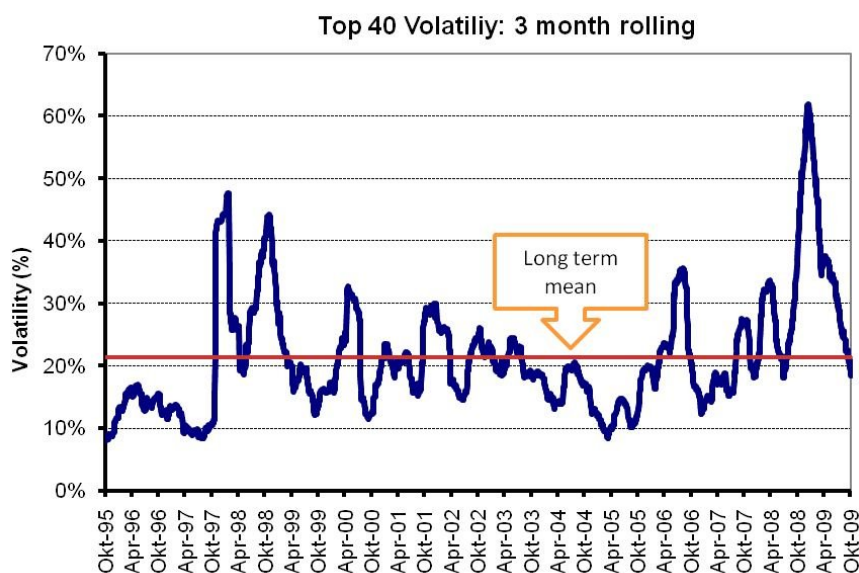


Figure 2: FTSE/JSE Top40 3 month historical volatility during the period, June 1995 through October 2009. The plot shows that volatility is not constant and seems to be stochastic in nature. Also evident is the phenomenon of mean reversion.

The stochastic nature of volatility led researchers to model the volatility surface in a stochastic framework. These models are useful because they explain in a self-consistent way, why options with different strikes and expirations have different Black & Scholes implied volatilities. Another feature is that they assume realistic dynamics for the underlying [Le 00]. The stochastic volatility models that we briefly appraise are the popular Heston model [He 93] and the well-known SABR model by Hagan et al. [HK 02].

In the Heston model, volatility is modelled as a long-term mean reverting process. The Heston model fits the long-term skews well, but fails at the shorter expirations [Ga 09]. Heston models the volatility surface as a joint dynamic in time and strike space. A viable alternative to the Heston model is the well known SABR model [We 05]. Here, volatility is modelled as a short term process by assuming that the underlying is some normally distributed variable. The SABR model assumes that strike and time to expiration dynamics are disjoint, i.e. the skews and term structure of the skews are calibrated separately. This model works better for shorter expirations but because volatilities do not mean revert in the SABR model, it is only good for short expirations. Another problem with the SABR model<sup>3</sup> is that its parameters are

<sup>3</sup>The SABR stochastic volatility model has the at-the-money volatility as input, which means that in an illiquid options market, the SABR generated market surface accuracy will have additional errors arising from the illiquidity of the at-the-money options trade data.

time-homogenous. This means that the model implies that future volatility surfaces will look like today’s surface. West has shown how to calibrate the SABR model using South African index option data [We 05]. Bosman et al. also showed how to obtain a representative South African volatility surface by implementing the SABR model using Alsi option data [BJM 08]. Issues with the calibration of the SABR model has been tackled by Tourrucô [To 08] while the original SABR formula has been rectified by Oblój [Ob 08].

A general problem associated with stochastic volatility models is that they fail to model the dynamics of the short term volatility skews. This arises from the fact that the at-the-money volatility term structure can be so intricate in the short-end<sup>4</sup> that these models just fail at accurately modelling the short end volatility dynamics. In fact, Jim Gatheral states [Ga 06]

*“So, sometimes it’s possible to fit the term structure of the at-the-money volatility with a stochastic volatility model, but it’s never possible to fit the term structure of the volatility skew for short expirations... a stochastic volatility model with time homogenous parameters cannot fit market prices!”*

The possibility of using an extended stochastic volatility model with correlated jumps in the index level and volatility might fit the short-term market volatility skews better, but in practice it is difficult (if not impossible) to calibrate such a complex model [Ga 09].

The above-mentioned problems associated with stochastic volatility models led us to look beyond these models.

## 4.2 Empirical Approaches

The empirical approach known as the Vanna-Volga method has been studied extensively. This approach was introduced by Lipton and McGhee [LM 02]. The Vanna-Volga method is also known as the traders’ rule of thumb. It is an empirical procedure that can be used to infer an implied-volatility smile from three available quotes for a given maturity; it is thus useful in illiquid markets. It is based on the construction of locally replicating portfolios whose associated hedging costs are added to corresponding Black-Scholes prices to produce smile-consistent values. Besides being intuitive and easy to implement, this procedure has a clear financial interpretation, which further supports its use in practice [Wy 08].

The Vanna-Volga approach considers an option price as a Black-Scholes price corrected by hedging costs caused by stochasticity in price-forming factors (volatility, interest rate, etc.) observed from real markets. While accounting for stochasticity in volatility, it differs from stochastic volatility frameworks (Heston, SABR, and modern

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<sup>4</sup>Intricate in the sense that the short term volatility skews can include volatility jumps that are correlated with the equity index level, that are usually not assumed when stochastic volatility models are derived to model the equity index volatility surface.

Levy process-based generalizations) in the following way: rather than constructing a parallel (possibly correlated) process for the instantaneous volatility and defining the price as the risk-neutral expectation, the Vanna-Volga approach puts much more weight on the self-financing argument, considering an option price as a value of the replicating Black-Scholes portfolio plus additional corrections offsetting stochasticity in volatility [Sh 08].

This approach is very popular in the foreign exchange market but has been applied to equities as well. Bosman et al. [BJM 08] implemented it with South African index data while Weizmann used Dow Jones Euro Stoxx 50 data [We 07].

### 4.3 Nonparametric Estimation of the Skew

In illiquid markets it might not be possible to calibrate models (stochastic or deterministic) due to a lack of data. In such circumstances nonparametric option pricing techniques might be feasible.

Nonparametric option pricing techniques utilise spot market observed security prices in order to determine the probability distribution of the underlying asset [Ja 05]. Using stock prices has the disadvantage of offering no guarantee of matching option prices. However, supposing that the model is right, we have a great quantity of data input for calibration.

Derman and Kani [DK 94] and Rubinstein [Ru 94] utilise implied binomial tree (IBT) approaches to find risk-neutral distributions which result in estimated option prices that match observed option prices. These nonparametric approaches impose no assumptions about the nature of the probability distribution of the asset. This leaves little room for misestimation of option pricing resulting from an incorrect choice of the probability distribution of the underlying asset. Thus nonparametric methods do not suffer from the smile bias that is evidenced in parametric models. Using nonparametric methods to estimate the price of an option and reversing this price through the Black-Scholes formula will yield the conventional implied volatility. Following this procedure for a range of possible strikes will result in the so-called nonparametric volatility skew. The nonparametric volatility skew is free of any model-specific assumptions and, as it is based only on observed asset data, it is a minimally subjective estimate of the volatility skew [AM 06].

An IBT is a generalization of the Cox, Ross, and Rubinstein binomial tree (CRR) for option pricing [Hu 03]. IBT techniques, like the CRR technique, build a binomial tree to describe the evolution of the values of an underlying asset. An IBT differs from CRR because the probabilities attached to outcomes in the tree are inferred from a collection of actual option prices, rather than simply deduced from the behavior of the underlying asset. These option implied risk-neutral probabilities (or alternatively, the closely related risk-neutral state-contingent claim prices) are then available to be used to price other options. Arnold, Crack and Schwartz showed how an IBT can be implemented in Excel [ACS 04].

De Araújo and Maré compared nonparametrically derived volatility skews to mar-

ket observed implied volatility skews in the South African market [AM 06]. Their method is based only on the price data of the underlying asset and does not require observed option price data to be calibrated. They characterised the risk-neutral distribution and density function directly from the FTSE/JSE Top40 index data. Using Monte Carlo simulation they generated option prices and found the implied volatility surface by inverting the Black-Scholes equation. This surface was found to closely resemble the Safex traded implied volatility surface.

## 5 The Deterministic Volatility Approach

Implementing stochastic volatility models and implied binomial trees can be very difficult. With stochastic volatility, option valuation generally requires a market price of risk parameter, which is difficult to estimate [DFW 98]. However, if the volatility is a deterministic function of the asset price and/or time, the estimation becomes a lot simpler. In this case it remains possible to value options based on the Black-Scholes partial differential equation although not by means of the Black-Scholes formula itself.

### 5.1 Deterministic Models

Deterministic volatility functions (DVF) are volatility models requiring no assumptions about the dynamics of the underlying index that generated the volatility. These models can only be implemented if one assumes that the observed prices of options reflect in an informationally efficient way everything that can be known about the true process driving volatility — the efficient market hypothesis [Re 06].

Deterministic volatility functions were introduced in the 1990s. During 1993, Shimko studied the risk neutral densities of option prices and chose a simple parabolic function as a possible parametric specification for the implied volatility [Sh 93]. Dumas et al. studied the applicability of such simple functions. They started by rewriting the general Black-Scholes differential equation as a forward partial differential equation (applicable to forward or futures contracts)

$$\frac{1}{2}\sigma^2(F, T) K^2 \frac{\partial^2 C}{\partial K^2} = \frac{\partial C}{\partial t}$$

with the associated initial condition,  $C(K, 0) \equiv \max(S - K, 0)$ . Also,  $S$  is the spot price,  $F$  the forward or futures price,  $K$  the absolute strike price,  $T$  is the time to expiration and  $C(K, T)$  is the call price. The advantage of the forward equation approach is that all option series with a common time to expiration can be valued simultaneously. They further mentioned that  $\sigma(K, T)$  is an arbitrary function. Due to this they posit a number of different structural forms (deterministic functions) for the implied volatility as a function of the strike and time to expiry [DFW 98]

$$\begin{aligned}
\text{Model 0 : } \sigma &= a_0 \\
\text{Model 1 : } \sigma(K) &= a_0 + a_1K + a_2K^2 \\
\text{Model 2 : } \sigma(K, T) &= a_0 + a_1K + a_2K^2 + a_3T + a_5KT \\
\text{Model 3 : } \sigma(K, T) &= a_0 + a_1K + a_2K^2 + a_3T + a_4T^2 + a_5KT
\end{aligned} \tag{1}$$

Here, the variables  $a_0$  through  $a_5$  are determined by fitting the functional forms to traded option data. Model 0 is the Black-Scholes model with a constant volatility where Model 1 attempts to capture variation with the asset price. Models 2 and 3 capture additional variation with respect to time. They estimated the volatility function  $\sigma(K, T)$  by fitting the functional forms in Eq. (1) to observed option prices at time  $t$  (today). They used S&P 500 index option data captured between June 1988 and December 1993. Estimation was done once a week. The parameters and thus skews were estimated by minimising the sum of squared errors of the observed option prices from the options' theoretical deterministic option values given by the functional forms<sup>5</sup>. Models 1 through 3 all reflected volatility skews or smirks. They found that the Root Mean Squared Valuation Error of Model 1 is half of that of the Black-Scholes constant volatility model. Another important result from their study is that the functional forms lose their predictive ability rather quickly. This means that these functional forms need to be refitted to traded data on a regular basis.

Beber followed the Dumas et al. study but used options written on the Mib30 — the Italian stock market index [Be 01]. He optimised two models

$$\begin{aligned}
\text{Model 1 : } \sigma &= \beta_0 + \beta_1K + \epsilon \\
\text{Model 2 : } \sigma &= \beta_0 + \beta_1K + \beta_2K^2 + \epsilon
\end{aligned} \tag{2}$$

One difference between his and the Dumas study is that he defined  $K$  as the moneyness and not absolute strike. Model 1 is linear in the moneyness and Model 2 quadratic i.e. it is a parabola.  $\epsilon$  is just an error estimate. The simplicity of the two models is determined by the endeavour to avoid overparametrization in order to gain better estimates' stability over time. Beber also decided to give the same weight to each observation, regardless the moneyness, as the strategy to assign less weight to the deep out of the money options owing to the higher volatility has not proven to be satisfactory. He fitted the traded option data by ordinary least squares. Beber found that the average implied volatility function is fitted rather well by a quadratic model with a negative coefficient of asymmetry. Hence the average risk neutral probability density function on the Italian stock market is fat tailed and negatively skewed. The interpretation of the parameters in general are as follows

- $\beta_0$  represents a general level of volatility which localizes the implied volatility function (it is also the constant of regression),

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<sup>5</sup>They used an algorithm based on the downhill simplex method of Nelder and Mead [PFTV92].

- $\beta_1$  characterizes the negative profile which is responsible for the asymmetry in the risk neutral probability density function; it is the coefficient that controls the displacement of the origin of the parabola with respect to the ATM options, and
- $\beta_2$  provides a certain degree of curvature in the implied volatility function or it controls the wideness of the smile.

One of the most comprehensive studies was done by Tompkins in 2001 [To 01]. He looked at 16 different options markets on financial futures comprising four asset classes: equities, foreign exchange, bonds and forward rate agreements. He compared the relative smile patterns or shapes across markets for options with the same time to expiration. He also used a data set comprising more than 10 years of option prices spanning 1986 to 1996. The individual equities examined were: S&P500 futures, FTSE futures, Nikkei Dow futures and DAX futures. He fitted a quadratic volatility function to the data and found his graphs of the implied volatility to be similar to that shown by Shimko in 1993 [Sh 93] and Dumas et al [DFW 98]. Tompkins then states:

*”If the sole objective was to fit a curved line, this has been achieved”.*

He concluded that regularities in implied volatility surfaces exist and are similar for the same asset classes even for different exchanges. A further result is that the shapes of the implied volatility surfaces are fairly stable over time.

Many studies followed the Dumas and Tompkins papers, most using different data sets. All of these studied the models listed in Eqs. (1) and (2). Sehgal and Vijayakumar studied S&P CNX Nifty index option data [SV 08]. These options trade on the derivatives segment of the National Stock Exchange of India<sup>6</sup>. Badshah used out-the-money options on the FTSE 100 index and found the quadratic model to be a good fit [Ba 09]. Zhang and Xiang also studied S&P500 index options and found the quadratic function fits the market implied volatility smirk very well. They used the trade data on 4 November 2003 for SPX options expiring on 21 November 2003 - thus very short dated options [ZX 05]. They used all out-the-money puts and calls and fitted the quadratic function by minimising the volume weighted mean squared error and found the quadratic function to work very well.

Another comprehensive study was done by Panayiotis et al. [PCS 08]. They tackled the deterministic methodology from a different angle. They considered 52 different functional forms to identify the best DVF estimation approach for modelling the implied volatility in order to price S&P500 index options. They started with functions as given by Dumas et al. and listed in Eq. (1) where  $K$  is the absolute strike. Next they changed the strike to  $\ln K$ ,  $S/K$  (moneyness) and lastly to  $\ln(S/K)$ . All in all they considered 16 functions similar to Eqs. (1). They also considered a

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<sup>6</sup>The S&P CNX Nifty is an index comprising the 50 largest and most liquid companies in India with about 60% of the total market capitalisation of the Indian stock market.

number of asymmetric DVF specifications. Their dataset covered the period January 1998 to August 2004 — 1675 trading days. They recalibrated all 52 functions on a daily basis. Their main result is that the deterministic specification with strike used as moneyness ( $S/K$ ) works best in-sample while the model with strike used as  $\ln K$  works best out-of-sample.

Modeling the volatility skew as a deterministic process has other benefits too [Bu 01]

- They allow one to model volatility separately in expiry time and strike. This means that each expiry's skew can be independently calibrated minimising compounding errors across expiries. This property is useful in modeling volatility surfaces in illiquid markets where data is sparse across strikes.
- Pricing using deterministic volatility preserves spread arbitrage market conditions because no assumptions about the underlying process is made.
- The whole surface can be calibrated with minimal model error.

## 5.2 Principle Component Analysis

Using a 3 parameter quadratic function was further motivated by the research findings due to Carol Alexander [Al 01]. She did a principle component analysis (PCA) on FTSE 100 index options and found that 90% of the dynamics of the volatility skew are driven by three factors

- parallel shifts (trends),
- tilts (slopes), and
- curvature (convexity).

Badshah also did a PCA on FTSE 100 options. He used the implied volatility surfaces for the March and October months for the years 2004, 2005, 2006 and 2007. His results are in line with that of Alexander although he found that on average 79% of the dynamics of the skews are driven by the first 3 components — this is similar to the findings by Alexander [Ba 09]. Le Roux studied options on the S&P500 index with strikes ranging from 50% to 150% [Le 07]. He found that 75.2% of the variation of the implied volatility surface can be described by the first principle component and another 15.6% by the second! His first component reflects the slope or tilt of the skew. The difference between his study and that of Alexander's is that he used moneyness instead of absolute strikes.

Bonney, Shannon and Uys followed Alexander's methodology and studied the principle components of the JSE/FTSE Top 40 index [BSU 08]. They found that the trend affect explains 42% of the variability in the skew changes, the slope 19% and the convexity an additional 14%. Their results show that the first three components

explain 76.24% of the variability in skew changes. They attribute the difference between Alexander’s 90% and their 76% to differences between a liquid market and less liquid emerging market.

### 5.3 The SVI Model

Another interesting research finding was presented by Jim Gatheral [Ga 04]. He derived the Stochastic Volatility Inspired (SVI) model. This is a 5 parameter quadratic model (in moneyness) based on the fact that many conventional parameterisations of the volatility surface are quadratic as discussed in §5.1. This parametrisation has a number of appealing properties, one of which is that it is relatively easy to eliminate calendar spread arbitrage. This model is “inspired” by the stochastic volatility models due to the fact that implied variance is linear in moneyness as  $K \rightarrow \pm\infty$  for stochastic volatility models. Any parametrisation of the implied variance surface that is consistent with stochastic volatility, needs to be linear in the wings and curved in the middle. The SVI and quadratic models exhibit such properties. Gatheral also states that:

*“if the wings are linear in strike (moneyness), we need 5 and only 5 parameters to cover all reasonable transformations of the volatility smile.”*

This model was extensively tested using S&P500 (SPX) index option data with excellent results.

### 5.4 The Quadratic Function for the ALSI Volatility Surface

The results from all the studies on deterministic volatility functions and the PCA studies mentioned above form the basis in modelling the South African ALSI index volatility surface. In scrutinising Eqs. (1) and (2) and taking illiquidity into account, we postulate that the following three parameter quadratic function should be a good model of fit for the ALSI implied volatility data (following the Beber notation in Eq. (2))

$$\sigma_{model}(\beta_0, \beta_1, \beta_2) = \beta_0 + \beta_1 K + \beta_2 K^2. \quad (3)$$

In this equation we have

- $K$  is the strike price in moneyness format (Strike/Spot),
- $\beta_0$  is the constant volatility (shift or trend) parameter,  $\beta_0 > 0$ . Note that

$$\sigma \xrightarrow{K \rightarrow 0} \beta_0,$$

- $\beta_1$  is the correlation (slope) term. This parameter accounts for the negative correlation between the underlying index and volatility. The no-spread-arbitrage condition requires that  $-1 < \beta_1 < 0$  and,



- $\beta_2$  is the volatility of volatility ('vol of vol' or curvature/convexity) parameter. The no-calendar-spread arbitrage convexity condition requires that  $\beta_2 > 0$ .

Note that Eq. (3) is also linear in the wings as  $K \rightarrow \pm\infty$ . In Fig. 5.4 we plot the volatility skew for the near Alsi as obtained by implementing Eq. (3). A discussion of

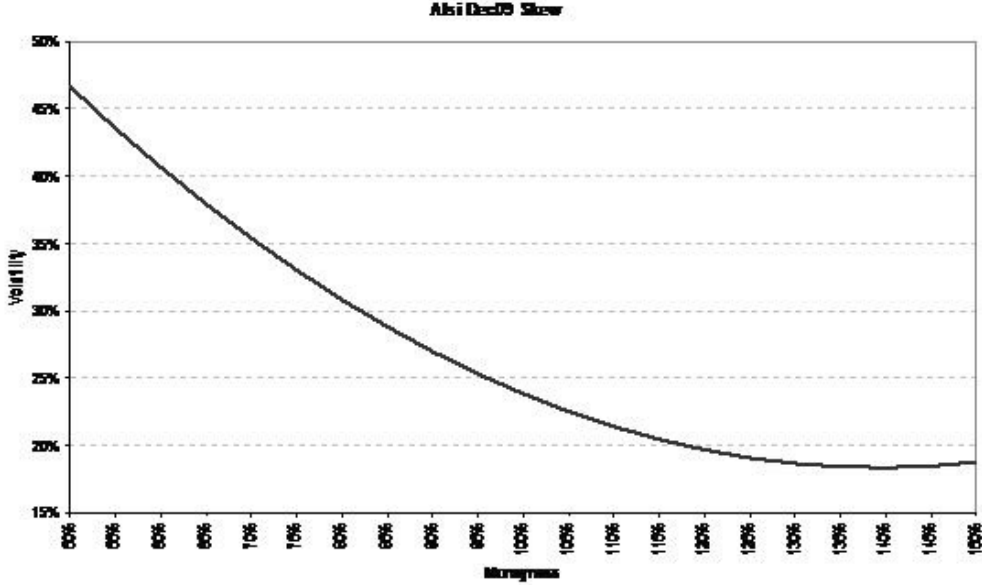


Figure 3: The ALSI volatility skew for the December futures contract at the beginning of October 2009.

the no-arbitrage parameter constraints on the correlation  $\beta_1$  and volatility of volatility  $\beta_2$  parameters can be found in Appendix A.

## 5.5 Volatility Term Structure

The functional form for the skew in Eq. (3) is given in terms of moneyness or in floating format (sticky delta format). This DVF does also not depend on time. The optimisation is done separately for each expiry date. In order to generate a whole implied volatility surface we also need a specification or functional form for the *at-the-money* (ATM) volatility term structure. It is, however, important to remember that the ATM volatility is intricately part of the skew. This means that the two optimisations (one for the skews and the other for the ATM volatilities) can not be done strickly separate from one another. Taking the ATM term structure together with each skew will give us the 3D implied volatility surface.

It is well-known that volatility is mean reverting; when volatility is high (low) the volatility term structure is downward (upward) sloping [Al 01, Ga 04]. This was shown for the JSE/FTSE Top 40 index in Fig. 2. We therefore postulate the following

functional form for the ATM volatility term structure

$$\sigma_{atm}(\tau) = \frac{\theta}{\tau^\lambda}. \quad (4)$$

Here we have

- $\tau$  is the months to expiry,
- $\lambda$  controls the overall slope of the ATM term structure;  $\lambda > 0$  implies a downward sloping ATM volatility term structure (this is plotted in Fig. 5.5), whilst a  $\lambda < 0$  implies an upward sloping ATM volatility term structure, and
- $\theta$  controls the short term ATM curvature.

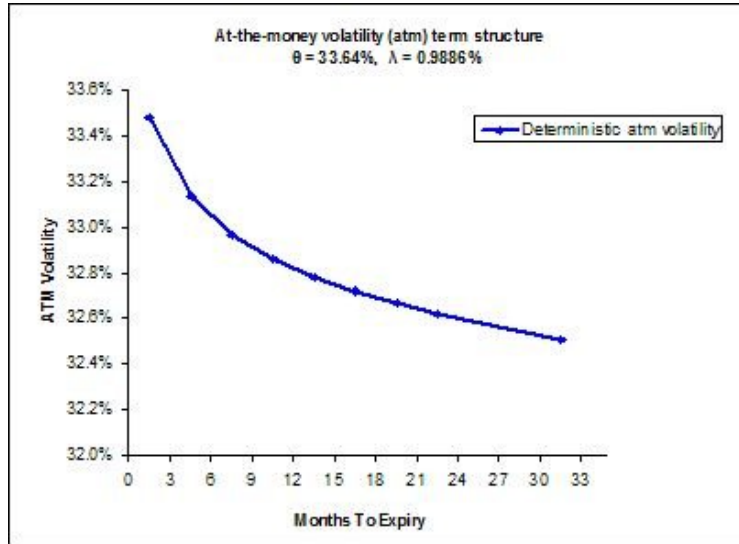


Figure 4: The market fitted at-the-money volatility term structure for the ALSI at the beginning of April 2009.

Please note that  $\tau$  is not the annual time to expiry. It actually is the “months to expiry”. It is calculated by

$$\frac{date_{expiry} - date_0}{365} * 12.$$

The reason for this notation is that if  $\tau = 1$ , the ATM volatility is given by  $\theta$ ; the 1 month volatility is thus just  $\theta$ . This makes a comparison between the South African 1 month volatility, and the 1 month volatilities offshore, like the VIX, easier.

A better understanding of these parameters is obtained if we consider the deterministic term of the Heston stochastic differential equation [He 93]

$$d\sigma(\tau) = \frac{\lambda}{\tau}(\omega - \sigma_I(\tau)) dt \quad (5)$$

where  $\omega$  is a long term mean volatility and  $\lambda/\tau$  is the mean reversion speed.

The solution to the ordinary differential equation in (5) is given by

$$\sigma(\tau) = \omega + \frac{\sigma_0 - \omega}{\tau^\lambda} \quad (6)$$

Comparing Eqs. (4) and (6) let us deduce that

- $\theta$  is a term that represents the difference between the current at-the-money volatility  $\sigma_0$  and the long term at-the-money volatility  $\omega$ , and
- $\lambda$  is a parameter defined such that  $\lambda/\tau$  is the mean reversion speed useful for ATM calendar spreads.

In using the volatility skew function given in Eq. (3) and the volatility term structure function shown in Eq. (4), we can generate the market equity index volatility surface in the deterministic framework.

## 6 Implementing the Deterministic Volatility Functions

Implementing the deterministic functions given in Eqs. (3) and (4) means we have to optimise or fit these functions to the trade data in a meaningful manner. In this section we show that estimating the volatility surface in the deterministic framework is a robust approach to modelling the equity index volatility surface.

### 6.1 The Data

Most index options are traded on the Alsi futures contracts. These contracts are listed on Safex. Still, the ALSI options market can be very illiquid at times with few, if any, at-the-money trades. The market is even more illiquid in long-term option trades. The number of trades differs substantially on a daily basis. There can be anything from as few as 5 near dated trades to more than 50 on any given day. Most trades are concentrated in the near and next-near contracts (3 to 6 months expiries). Currently these are for the Dec09 and Mar10 contracts. A good day will also have a good number of trades in the next contract i.e., Jun10 (9 month expiry). Options on futures with more than 9 months to expiry hardly ever trade.

Before any calibration is done, the data needs to be cleaned. The raw trade data is extracted from the Nutron trading system<sup>7</sup>. The trade data includes the trade date, futures spot, strike, traded volatility, option type, and the volumes traded. In

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<sup>7</sup>This trading system is used by all members and were developed by Securities and Trading Technology (<http://www.tsti.co.za/>). Safex supplies the system free of charge to all derivatives members.

general the average number of contracts traded is 1000 per single trade. Single trades with less than 10 contracts are discarded.

The sparseness of market data makes the volatility calibration prone to significant model errors. This holds especially if the market implied skew and the term structures are estimated jointly with the risk of errors being compounded. In this light, the deterministic model is appropriate as it models the volatility surface separately in skews (strikes) and in term structures (expiry times). This approach is suitable for modelling volatility surfaces using sparse option trades, because it indirectly accommodates for the fact that the relationship of volatility on the strikes and through time can change independently with different statistical errors.

Note, options with expiries of less than a month have parameter t-statistics (see Appendix B) that are unreasonably high. This is probably the case because the very short end of the surface is prone to volatility jumps. To preserve the volatility dynamics (no-arbitrage bounds), the very short time to expiry trades (usually < 1 month) are omitted in the deterministic volatility construction.

## 6.2 Calibrating the Model Skews

The volatility surface is obtained by calibrating the skews using the functional form in Eq. (3) and the empirical data — this means we fit the data to the given equation. The minimization problem we have to solve at time  $t_0$  is stated as

$$\omega_i \min_{\beta_k} \left\| \sigma_{t_i}^{model} - \sigma_{t_i}^{traded} \right\|^2 \quad \text{with } t_i \in [t_0 - h, t_0]; k = 1, 2, 3 \quad (7)$$

subject to the constraints;  $\beta_0, \beta_2 > 0$ , and  $-1 < \beta_1 < 0$ . Here  $h \sim 7$  and  $\omega_i$  are weights such that the optimisation are biased towards the most recent traded data.

The optimisation is performed using the Nelder-Mead grid search [RV 07], [PFTV92]. The average error is monitored to ensure it remains within an tolerable limit. On any given day, there might not be enough trades to afford a good fit. In such situations we use all trades from the last 7 days bulked together. This naturally suggests that we employ a time decay weighting scheme. To this end we use an exponentially weighted moving average, with the decay constant adjusted for time. Note that this minimisation will give us the floating skews which entails that we fit the data to obtain the shape of the skews only. A broader discussion on the fitting procedure is given in Appendix C.

Fig. 6.2 shows a typical ALSI skew calibration. We show the scattered data together with the fitted function from Eq. (3). This calibration was done on 7 October 2009 and the parameters were found to be:  $\beta_1 = -0.52$ ,  $\beta_2 = 0.10$  and  $\beta_0 = 0.65$ . The error associated with the skew calibration is usually less than 1.5%; in this case it was 0.517%. The parameters are calculated for each expiry. We list the optimised parameters at the beginning of October 2009 in Table 6.2.

Eq. (3) becomes useful if we realise that the ATM volatilities have a moneyness of 100%. By substituting this back into Eq. (3) we obtain the model ATM volatility

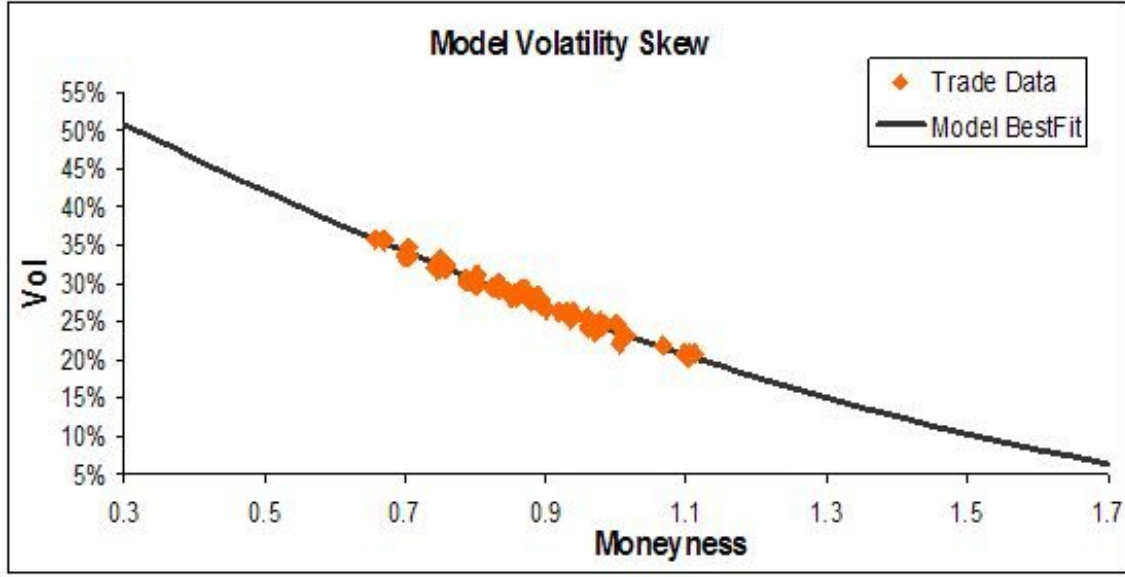


Figure 5: The Mar10 Alsi volatility skew as obtained by optimising and calibrating the deterministic volatility model. The average calibration error is 0.517%.

as

$$\sigma_{atm}^{model}(\beta_0, \beta_1, \beta_2, \tau) = \beta_0 + \beta_1 + \beta_2. \quad (8)$$

In using Eqs. (3) and (8) we obtain the model floating skew as follows

$$\begin{aligned} \sigma_{float}^{model}(\tau) &= \sigma_{model} - \sigma_{atm}^{model} \\ &= \beta_1(K - 1) + \beta_2(K^2 - 1). \end{aligned} \quad (9)$$

Eq. (9) can now be implemented to obtain the correct relative/floating volatility skews. Fig. 6.2 shows some of the floating skews.

Expiry	Slope ( $\beta_1$ )	Shift ( $\beta_0$ )	VolVol ( $\beta_2$ )
17-Dec-09	-0.78689926	0.24831875	0.77464525
18-Mar-10	-0.68597081	0.20341598	0.71784505
17-Jun-10	-0.63671087	0.18253990	0.68877327
16-Sep-10	-0.60478572	0.16939262	0.66939456
15-Dec-10	-0.58168726	0.16007358	0.65508826
17-Mar-11	-0.56323816	0.15274954	0.64347895
15-Dec-11	-0.52509543	0.13795182	0.61892680

Table 1: Optimised parameters for Eq. (3) using actual trade data as supplied by Safex. Optimisation was done at the beginning of October 2009.

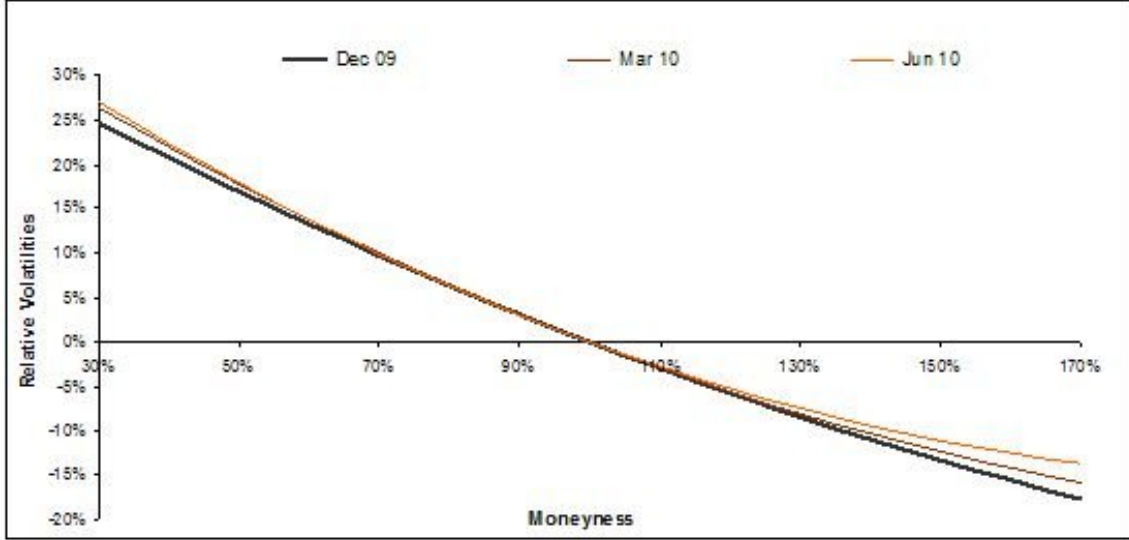


Figure 6: The deterministic floating skews for three different expiries.

### 6.3 Calibrating the Term Structure of at-the-money Volatilities

Eq. (9) gives us the floating volatilities only whilst Eq. (8) gives us the model ATM volatilities. To obtain the correct absolute volatilities, we need the correct ATM volatilities. This is now done by separately calibrating Eq. (4). As a basis for the at-the-money volatility term structure determination, we utilize the at-the-money volatilities defined by the optimised skews and given in Eq. (8). We then minimise the function

$$\min_{\theta, \lambda} \left\| \sigma_{atm}^{model}(\tau) - \frac{\theta}{\tau^\lambda} \right\|^2 \quad (10)$$

with  $\tau$  the months to expiration. A full description of the optimisation is given in Appendix D.

The parameters obtained in optimising Eq. (3) are usually very accurate for the first 3 expiries due to enough trade data available. This entails that the optimisation of Eq. (4) is also quite good. However, if there is only enough data available in the first two expiries, the fitting error will be larger. In Table 6.3 we list the optimised parameters as well as ATM volatilities obtained at the beginning of October 2009.

The current term structure is plotted in Fig. 7. Note that the errors are higher for the very short and long times to expiration. This is because the very short end of the ATM volatility curve is prone to volatility jump risk and is therefore difficult to model [Ga 06]. The long end suffers from the sparse data problem.

By combining the estimated ATM volatility term structure and the relative volatility skews, the volatility surface is obtained.

Date	Months to Expiry	ATM Vols	ATM	
17-Dec-2009	2.367123288	24.882488574	$\lambda$	0.012166143
18-Mar-2010	5.358904110	24.636363040	$\theta$	0.251447104
17-Jun-2010	8.350684932	24.503765914		
16-Sep-2010	11.342465753	24.412649496		
15-Dec-2010	14.301369863	24.343899598		
17-Mar-2011	17.326027397	24.287144045		
16-Jun-2011	20.317808219	24.240123090		
15-Sep-2011	23.309589041	24.199646164		
15-Dec-2011	26.301369863	24.164119665		

Table 2: The ATM term structure of volatility and parameters obtained by calibrating Eq. (4).

## 6.4 Robustness of the Model

It is a well-known fact that volatility surfaces generated from models need to be stable in order to achieve reliable valuations and sound risk management calculations. In this light, the deterministic volatility model parameters are constrained and it can be shown that these parameters are unique.

If a model volatility skew is generated on a daily basis, one runs the risk that the wings can vibrate. The more illiquid the market, the more this becomes a problem. This is a consequence of the optimisation procedures employed and is obviously not how the markets trade on a daily basis. In order to overcome this problem we double smooth the skews and this compromise the reaction speed.

## 7 Tests and further Results

The model surface has been tested against the official Safex surface over the past 10 months. Up until the 7 October 2009, Safex's official surface was a polled surface. The market was polled on the first Monday of every month. On average about 6 to 9 market participants contributed their surfaces — most of these were market makers. The average surface was calculated and that surface was used as the official Safex surface from the Thursday following the 1st trading Monday. This surface was stale for the next month until the next polling Monday.

In Figs. 8, 9 and 10 we plot the model skews and polled skews for the Dec09, Mar10 and Jun10 option contracts. The polled skews were obtained on 5 October 2009. Similar results were obtained for all tests done since January 2009. In comparing the polled and model skews it is useful to realise that the polled surface was an equally weighted surface. Some contributors hardly ever traded but their surfaces had the same weight as those who traded the most. The model surface, on the other hand,

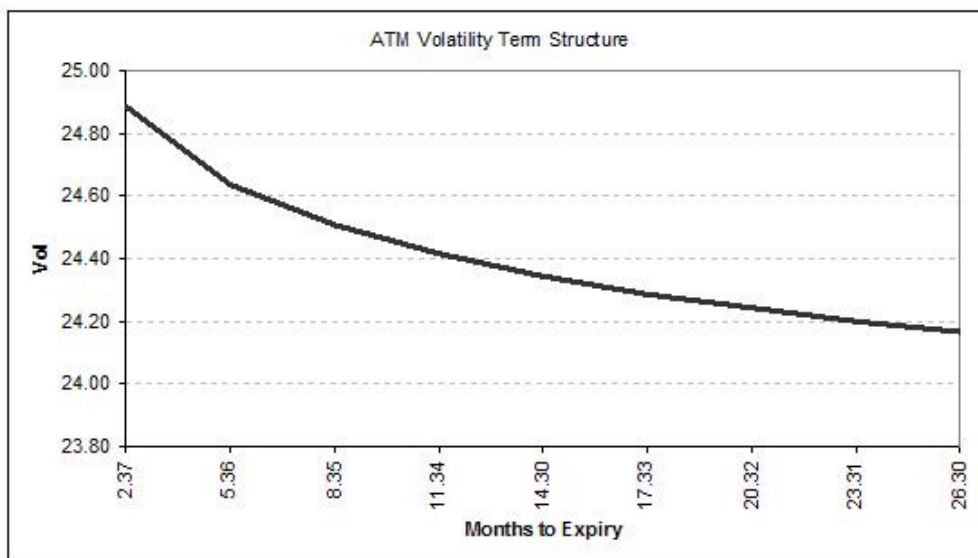


Figure 7: The ATM term structure of volatility at the beginning of October 2009.

will be biased towards the most active market player. In theory, if only one player trade actively, the model should reproduce that player’s surface!

The model surface is also consistent with the polled surface in that it reflects the no-spread-arbitrage rule — none of the skews crosses (see Appendix A).

These results show that the model surface is indeed a true market related surface. The model surface is also unbiased because actual traded data was used in its construction. It has been very robust and its dynamics are correlated to the dynamics of the underlying future contracts. The fitting methodology can be seen as a best-decency fit similar to fitting a yield curve to bond data. This means that we might not be able to price all options back to their original premiums but the skews are smooth and still arbitrage free.

## 8 Conclusion and Discussion

The deterministic model has been tested with market data over the last 10 months. The results are excellent and the model is very robust. These results show that the quadratic deterministic volatility function given in Eq. (3) is a good model to use as a specification of the implied volatility surface. A further analysis on the stability of the deterministic volatility function led to similar results as obtained by Tomkins [To 01]. We also found that the shape of the implied volatility surface is fairly stable over time. This is good news for the risk management processes employed by clearing houses.

With the optimisation we ensured that the surface is calendar-spread arbitrage free. Human interventions are minimised and no trading biases are incorporated. The volatility surface obtained with the model reflects the true traded volatilities and can thus be seen as the true traded implied volatility surface for options traded on the



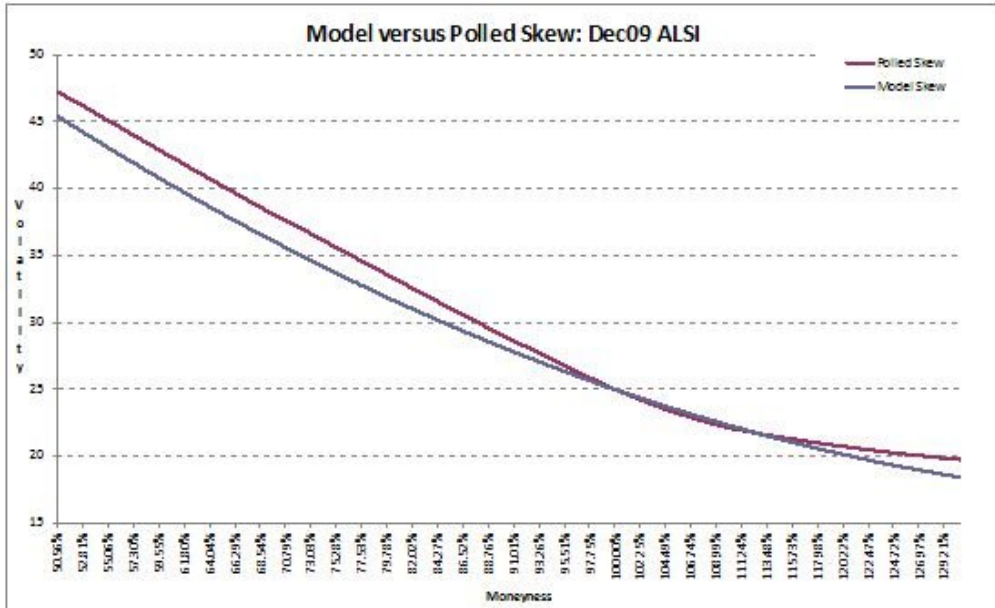


Figure 8: Comparison of the model and polled skew for Dec09 ALSI.

ALSI futures contracts.

## A Sufficient Conditions for no Spread Arbitrage

Arbitrage is a trading strategy that takes advantage of two or more securities being mispriced relative to each other. No-arbitrage conditions have been pivotal in the development of “fair” prices for all derivatives, especially in the futures and options market [16]. The aim of this annexure is to show that the constraints imposed on the parameters of the deterministic model, implies that the volatility surface should be arbitrage free [Ca 04, Ga 04, Le 04].

### A.1 Spread Arbitrage

We will now show that imposing put and call spread no-arbitrage constraints on the parameters implies a spread arbitrage free volatility surface when constructing the surface in the deterministic framework.

Suppose we construct the floating skews in the deterministic framework using the relation as defined by equation (3)

$$\sigma(K) = \beta_0 + \beta_1 K + \beta_2 K^2 \quad \text{with } \beta_0 > 0.$$

For equity index skews it is well known that the out-the-money puts, must trade at a higher volatility than out-the-money calls<sup>8</sup> i.e.

$$\frac{\partial \sigma(K)}{\partial K} < 0$$

<sup>8</sup>We limit ourselves to out-the-money options, because in general they are more liquid.

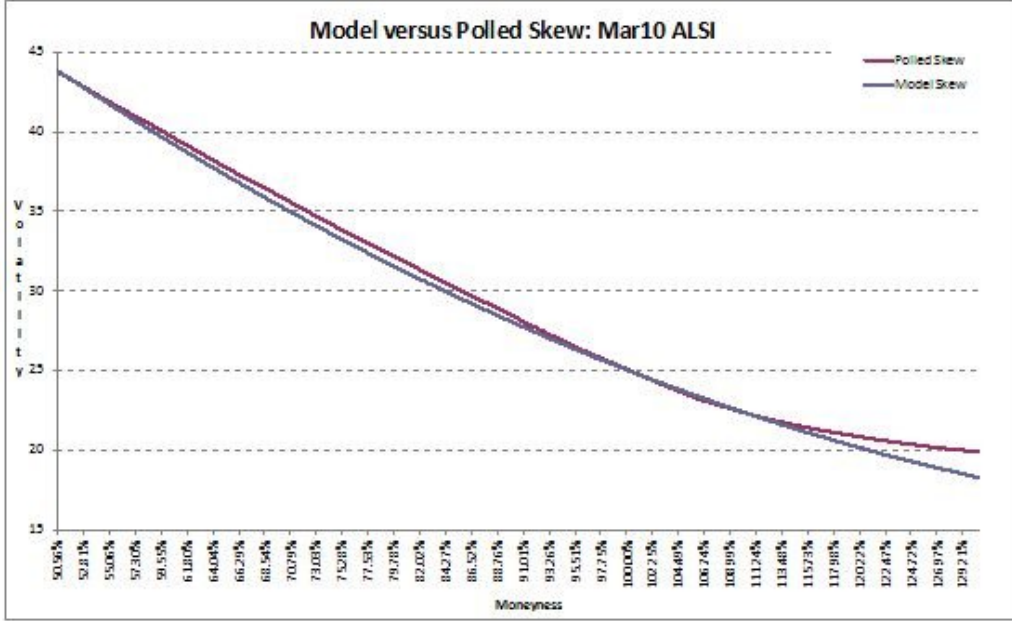


Figure 9: Comparison of the model and polled skew for Mar10 ALSI.

or

$$\beta_1 + 2\beta_2 K < 0. \quad (11)$$

From a trade perspective, the more out-the-money an option, the more convexity the skew should reflect [Ga 04]. This condition implies that

$$\frac{\partial^2 \sigma(K)}{\partial K^2} > 0$$

or equivalently that

$$2\beta_2 K > 0. \quad (12)$$

However, the moneyiness  $K > 0$  which means that in order for Eqs. (11) and (12) to hold we must have

$$\begin{aligned} -1 < \beta_1 < 0 \\ \beta_2 > 0. \end{aligned}$$

These conditions are imposed on the parameters when deriving the floating skews and leads to skews with no vertical (bull and bear) spread arbitrage. In addition to these, no other vertical spread arbitrage free constraints are used when constructing the floating skews when each of the asymptotic term structures of these parameters are derived [Ca 04].

In Fig. 11 we show the near and far expiries' floating skews obtained in a calendar spread arbitrage setting while, in Fig. 12 we show the near and far expiries' floating skews obtained in a no-calendar spread arbitrage setting.

The consequences of using the term structure of the vertical spread arbitrage free parameters to generate the market volatility surface are

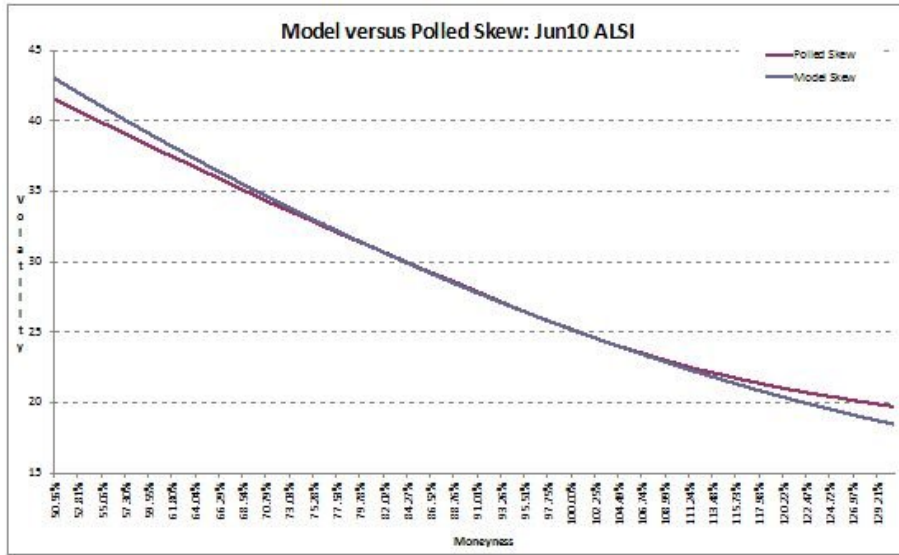


Figure 10: Comparison of the model and polled skew for Jun10 ALSI.

1. No two parameter estimates are the same, the parameters are unique.
2. None of the derived skew lines crosses, in other words, the final volatility surface is calendar spread arbitrage free.

Constraining and deriving the parameters in this way, ensures that the final volatility surface is arbitrage free for most spread strategies (bull, bear, and calendar spreads).

## A.2 Importance of Skew Parameter Constraints

The most fundamental spread strategy is the bull call spread. This spread is created by buying a call option on the index with a certain strike level, and then selling a call option on the same index, with a higher strike level. Both options expires on the same date. If the volatility is invariant to both options, we can expect that the price of a call will always decrease as the strike price increases. This means that the value of the option sold is always less than the value of the option bought. Thus with a flat volatility surface, consistent and fair pricing of the bull call spread requires an initial investment into the strategy.

In the real world, the volatility surface is not flat. The price of a bull call spread, is therefore not only a function of the strike level, but also a function of the volatility skew. The equity index implied volatility is in general a skewed curve in strike prices; such that call equity options with higher strike prices (out-the-money calls) have lower volatilities, than call equity options with lower strike prices (in-the-money calls)<sup>9</sup>. Thus in the presence of the equity volatility skew, the bull call spread, requires the

<sup>9</sup>Consistent and fair pricing of options requires that out-the-money calls must have a lower volatility than the more dearer in-the-money call option volatilities.

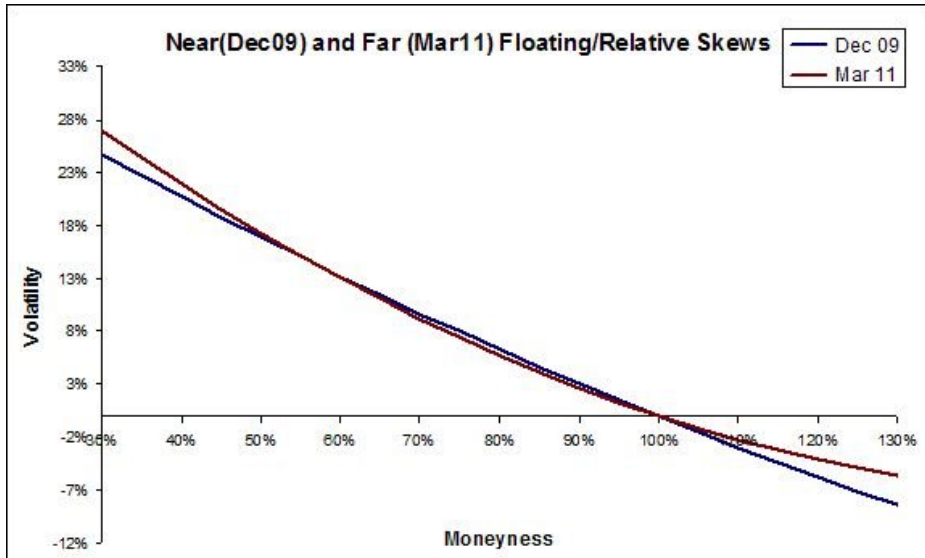


Figure 11: The near and far expiries' floating skews obtained in a calendar spread arbitrage setting.

buying of a call at a low strike, high volatility and selling a call at a high strike, lower volatility. In the presence of a volatility skew, consistent and fair pricing of the bull call spread requires an initial investment into the strategy.

### A.3 The Skew and the Arbitrageur

Suppose the volatility skew is skewed such that the lower call strike has lower volatility than the higher strike call option. An arbitrageur will then sell the expensive high strike call to fund the cheaper low strike call. But consistent and fair pricing of the bull call spread requires that the volatility must be skewed such that the lower call strike has higher volatility than the higher call strike. The volatility skew used by the arbitrageur, initially, must revert to the volatility skew used for fair and consistent pricing. When the volatility skew has reverted, the arbitrageur will close out his position and realise a risk-less profit.

## B Parameter Confidence Measure: t-statistic

In order, to determine which expiry specific parameters are statistically insignificant, we use the t-statistic [Al 01, Wi 09]. In short, the t-statistic measures the confidence associated with a particular parameter. The t-statistic is defined as the ratio of the value of the parameter to its standard error. Thus, the greater (lower) the t-statistic the higher (lower) the confidence associated with a particular parameter.

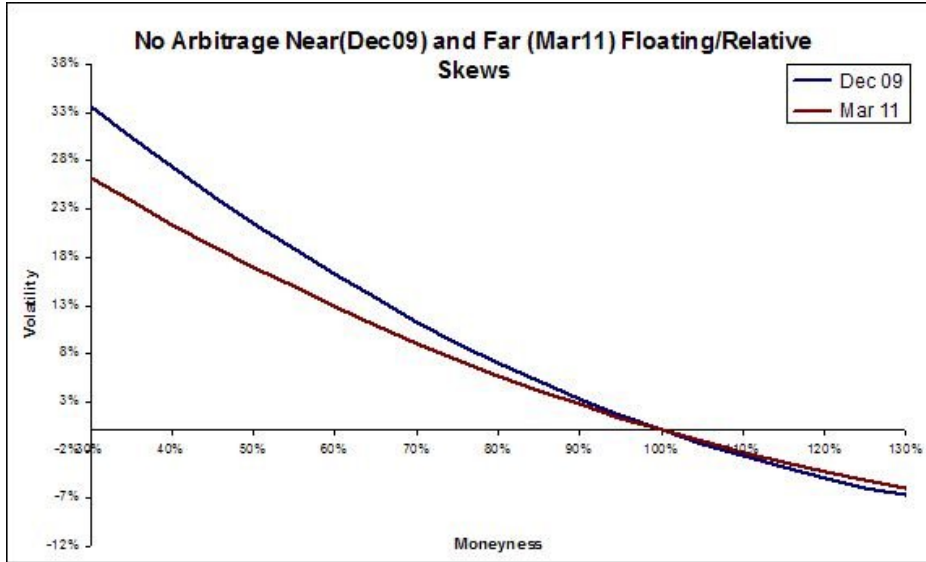


Figure 12: The near and far expiries' floating skews obtained in a no-calendar spread arbitrage setting.

Parameter confidence<sup>10</sup> is important for indicating whether the parameters are statistically sound. The parameter confidence statistic in this way provides a way of measuring whether the optimised parameters do define a statistically confident skew. This measure is especially important for very near expiries, where the volatilities are highly prone to jump risk — a difficult to model phenomena. A critical t-statistic value of 1% was used, which implies that, when optimising the volvol parameter and it has a t-statistic greater than 1%, it will be statistical significant and should be used as the final parameter (see Appendix A).

In Fig. 13 we plot the t-statistics for the parameters in Eq. (3). The t-statistic of all the parameters reaches a minimum when we are 1 to 1 $\frac{1}{2}$  months away from expiry. Since the t-statistic measures the confidence associated with the parameter, it is clear that the confidence of the parameters drops significantly for construction of the volatility surface for dates between 1 to 1 $\frac{1}{2}$  months to expiration.

## C Optimising the Deterministic Volatility Function

The objective function that is optimized for constructing the volatility surface using the AlsI option market data are formulated as follows: starting with a set of initial values for the parameters  $\beta_0$ ,  $\beta_1$  and  $\beta_2$  in the deterministic volatility setting, we find

<sup>10</sup>Volatility of volatility ( $\beta_2$ ) is as important to volatility traders as the volatility of an equity is to equity traders.

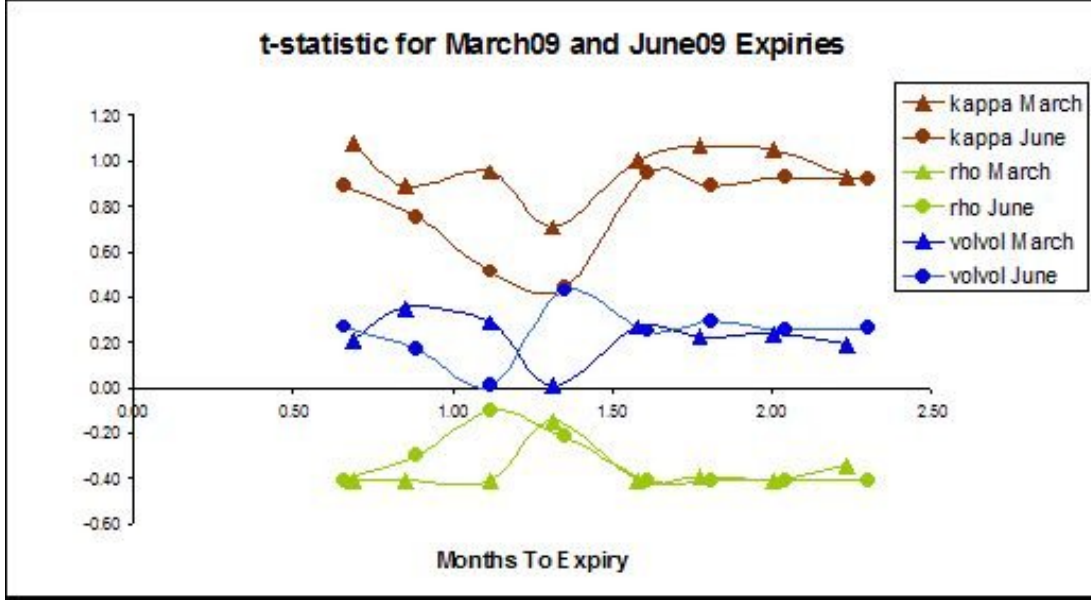


Figure 13: t-statistic of the deterministic model parameters for various months to the 18-March-09 and the 18-June-09 expiry.

the parameter triplet<sup>11</sup> such that the Euclidean distance between the traded implied volatility  $\sigma_{t_i}^{traded}$ , and the deterministic model volatility  $\sigma_{t_i}^{model}$  are minimised. The minimization problem we have to solve at time  $t_0$  is stated as

$$\omega_i \min_{\beta_0, \beta_1, \beta_2} \left\| \sigma_{t_i}^{model} - \sigma_{t_i}^{traded} \right\|^2 \quad \text{with } t_i \in [t_0 - h, t_0] \quad (13)$$

subject to the constraints;  $\beta_0, \beta_2 > 0$ , and  $-1 < \beta_1 < 0$ . Here  $h \sim 7$  and  $\omega_i$  are weights such that the optimisation is biased towards the most recent traded data.

The optimization is performed using the Nelder-Mead grid search, and the average error are monitored to ensure it remains within a tolerable limit<sup>12</sup> [PFTV92]. Note that *due to the sparseness of the market data in strikes, on some days there are not enough strikes per expiry to derive a statistically confident skew.*

During the optimisation the following holds

- The set of times  $[t_0 - h, t_0]$  for skew construction is defined such that it covers a wide enough range of strikes for confident skew construction. With the Alsi data, it was empirically established that  $h \sim 7$  trading days is optimal.
- The weights  $\omega_i$  are defined, such that the optimised parameter triplet set has accuracy that are biased towards the most recent traded volatilities. The weight-

<sup>11</sup>We conform to referring to the 3 parameters as the parameter set triplet, given that these parameters describes the dynamics of the volatility surface jointly. This is particularly so because the surface, at any given time, cannot be fully characterised by one of the parameters only.

<sup>12</sup>When the root mean square error are higher than 1.5% the data is either filtered for volatility outliers, or the weighting scheme is adjusted. Convergence with the Nelder Mead optimization are quick (with any arbitrary chosen initial parameter triplet set), when the data and weighting scheme are well defined.

ing scheme<sup>13</sup> used to effectively reflect the most recent market volatility skews are

$$\omega_i = 1 - \frac{1}{7}(1 - \lambda)(t_0 - t_i)$$

Through empirical tests we established that an optimal<sup>14</sup>  $\lambda$  is 91.5%. The optimum parameter triplet is then obtained by weighting yesterday's volatilities about 1% less than today's and letting all other historic volatilities recursively weigh 1% less per day. In this way, the optimised parameter triplet found that are used for deriving the model volatility skews accurately reflects the current market volatility skew levels<sup>15</sup>. An example of actual trade data and the weighting scheme is shown in Fig. 14.

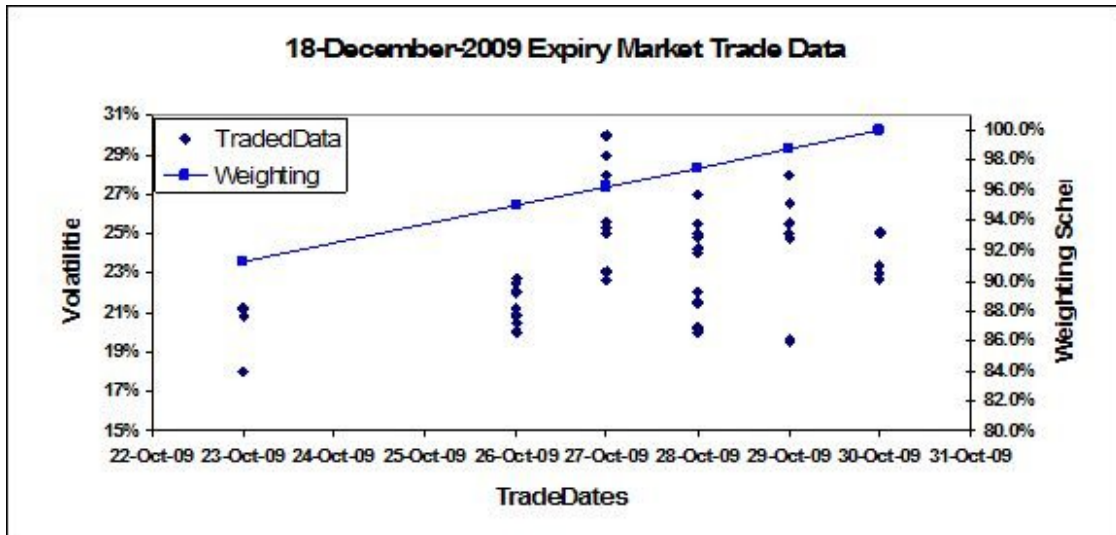


Figure 14: Traded volatilities of Alsi options expiring on 18-Dec-2009 showing sparse data. The weighting scheme used in the optimisation routine is also plotted.

<sup>13</sup>Note that this is a specific type of time decaying weighting scheme. The standard (exponential) volatility time weighing postulates that volatility is a lamda weighted linear sum of reaction and persistence terms. The standard exponential weighing scheme can therefore be said to be reactive to the current market volatility regime, yet capturing the important persistence characteristic of volatility. The standard exponential moving average, as eloquent as it is, applied to sparse volatility trade data, will distort the structure of the market volatility, especially because the trades are very sparse in strikes per day. For this reason we conform to use another time weighting average.

<sup>14</sup>Optimal in the sense of the time taken for the Nelder-Mead optimisation routine to find the optimal solution to the objective function in Eq. (13).

<sup>15</sup>In fact, this timeous response of the volatility skew is a necessary condition required for accurately modelling the effect of market shocks on the volatility skew.

## D Calibrating the At-The-Money Volatility Term Structure

### D.1 Obtaining the Short End

The optimisation of the at-the-money volatilities is very intricate and important for it determines the confidence associated with the estimation of the most important volatilities on the volatility surface i.e. the at-the-volatilities. In this light, for accurate estimation of the at-the-money term structure in an illiquid market setting, we first consider some important facts about the skew optimisation that will be useful for the at-the-money volatility estimation.

The optimized parameter triplet ( $\beta_0$ ,  $\beta_1$  and  $\beta_2$ ) obtained from the skew optimisation described in Appendix C cannot be used outright to determine the current at-the-money volatilities because

- The parameter triplet defines the skew over a certain data period  $[t_0 - h, t_0]$  that defines an at-the-money for the data period  $[t_0 - h, t_0]$ .
- If there are more at-the-money volatility trades than trades used in the optimisation for skews on any other day in the daily data set for the period  $[t_0 - h, t_0]$  using the optimised parameter triplet for skews can result in a mis-estimation of the current traded at-the-money volatility<sup>16</sup>. An example of this situation is depicted in Fig. 15.

The at-the-money term structure and the skews however have a common property: they jointly<sup>17</sup> define the volatility surface and its dynamics. Hence, as a basis for the at-the-money volatility term structure determination, we utilize the at-the-money volatilities defined by the optimised skews and given in Eq. (8). We then minimise the function

$$\min_{\theta, \lambda} \left\| \sigma_{atm}^{model}(\tau) - \frac{\theta}{\tau\lambda} \right\|^2 \quad (14)$$

with  $\tau$  the months to expiration.

This optimization is done using the Nelder-Mead grid search, and the results of the optimisation are the optimized  $\theta$  and  $\lambda$  parameters that define the structure of the at-the-money volatility term structure consistent with the dynamics of the skews [PFTV92]. However, note that these two parameters do not necessarily<sup>18</sup> define the parameters for the current at-the-money volatility term structure. To update the

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<sup>16</sup>More at-the-money trades than skew trades have not yet been observed amongst the Alsi data. It is in fact, the agricultural market that exhibits this volatility trading pattern.

<sup>17</sup>Jointly, in the sense that the at-the-money volatility term structure changes are usually followed by changes in the skews.

<sup>18</sup>When there is a range of at-the-money trades on a particular day in the skew trade day set, these parameters will mis-estimate the current at-the-money volatilities. Fortunately this rarely happens in the Alsi options market.



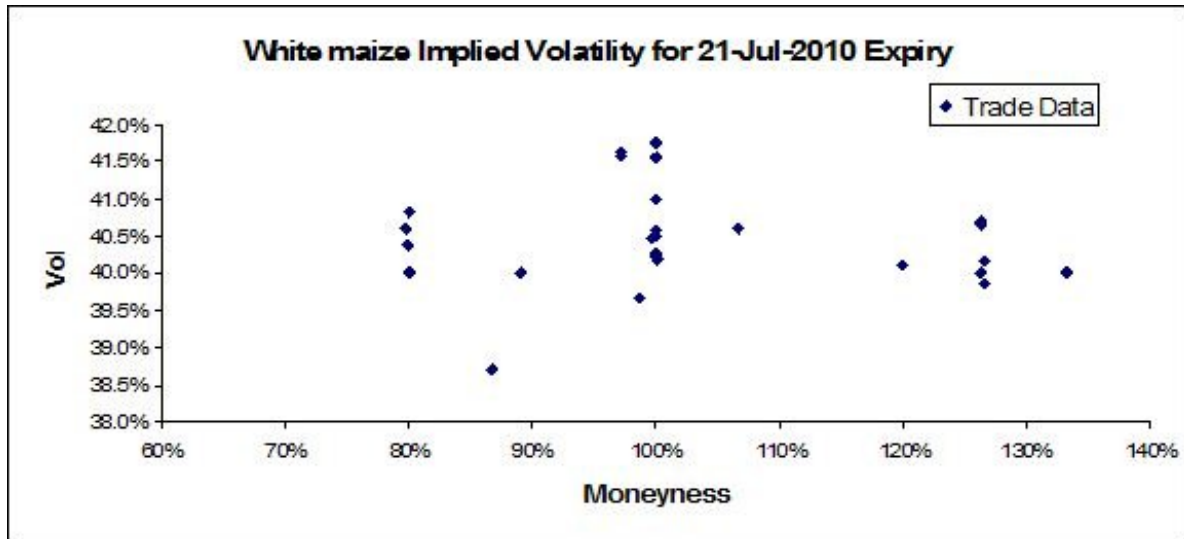


Figure 15: Traded volatilities of White maize options expiring on 21-Jul-2010. Multiple skew curve occurs because there are more at-the-money trades than well-defined trades around the at-the-money.

at-the-money volatility term structure to reflect the current at-the-money volatilities, a ridge factor,  $R$  is introduced. The final at-the-money volatility for any number of months to expiry  $\tau$ , is then determined using,

$$\sigma_{atm}^{model}(\tau) = \frac{\theta + R}{\tau^\lambda}. \quad (15)$$

The ridge factor is usually close to 1%. The idea with the ridge factor is to align the short end of the at-the-money volatility term structure with the current traded at-the-money volatilities and still keeping the long end of the term structure (as defined by the skews) intact. At present, no significant trades have been observed that traded exactly at-the-money, hence the ridge factor is zero at present. Given the time weighting of the skew optimisation and the current structure of the traded volatilities, a ridge factor of zero results in at-the-money volatilities that reflect the current level of the traded at-the-money volatilities correctly. The structure of the at-the-money trades are monitored daily and the ridge factor will be adjusted when necessary.

## D.2 Optaining the Far End

The rationale behind doing an optimisation to define the parameter term structures lies in the fact that there are very few Alsi option trades for expirations further than 9 months. The volatility skews cannot be determined due to a lack of data! To address this problem we studied the term structures of the deterministic parameter set that defines the polled volatility surface. The polled volatility surface included

the skews for far dated expiries as well. We found empirically that the parameter set (especially the ‘vol of vol’ parameter  $\beta_2$ ) exhibits a decaying term structure. We therefore decided to use the near dated expiries’ parameters  $\beta_k(\tau)$  (found in §C) to infer the far dated expirations via the minimization of

$$\min_{\theta_k, \lambda_k} \left\| \beta_k(\tau) - \frac{\theta_k}{\tau^{\lambda_k}} \right\|^2$$

for each parameter  $k$  of the triplet  $(\beta_0, \beta_1$  and  $\beta_2)$ .

The optimisation is performed using the Nelder-Mead grid search and each pair of optimised  $(\theta_k, \lambda_k)$  is used to determine the skews that define the far maturity end of the volatility surface. Due to the time decaying property of the parameter set, the expression used to determine the parameter term structure for skew construction, is

$$\beta_k^\tau(\theta_k, \lambda_k) = \frac{\theta_k}{\tau^{\lambda_k}}$$

where  $\tau$  is the months to expiration, and  $k$  is one of parameters in the deterministic parameter triplet  $(\beta_0, \beta_1, \beta_2)$ .

It is noted here that each pair  $(\theta_k, \lambda_k)$  that characterises the term structure of the skew parameters implies that the parameters mean-revert (Heston surface dynamic) which is qualitatively well in agreement with the perceived dynamics of volatility surfaces.

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